Question One: (15 Marks) *Start a new sheet of paper.*

a) Find $\int \cos^3 x dx$. [2]

b) Find $\int \frac{dx}{\sqrt{x^2+2x+5}}$. [2]

c) Evaluate $\int_1^2 x \ln x dx$ (in exact form). [3]

d)

i) Find real numbers a, b and c such that
$\frac{1}{(x^2+1)(x+1)} = \frac{ax+b}{(x^2+1)} + \frac{c}{(x+1)}$. [2]

ii) Hence evaluate $\int_0^1 \frac{1}{(x^2+1)(x+1)} dx$ (in exact form). [2]

e) Use the substitution $x = \sin^2 \theta$ to evaluate $\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}} dx$ [4]

Question Two: (15 Marks) *Start a new sheet of paper.*

a) Given that $z = \sqrt{3} + \frac{1+i}{1-i}$ find:

i) $\operatorname{Im}(z)$ [1]

ii) $\bar{z}$ [1]

iii) z in mod/arg form. [2]

b) Solve $z^2 = 3-4i$. [2]

c) Illustrate on an Argand diagram the region given by
$\left\{ z : 0 \le \arg(z+4+i) \le \frac{2\pi}{3} \, and \, |z+4+i| \le 4 \right\}$. [3]

(Question 2 continued over)

d) z is a point on the circle $|z-1|=1$ and $\arg(z)=\theta$.

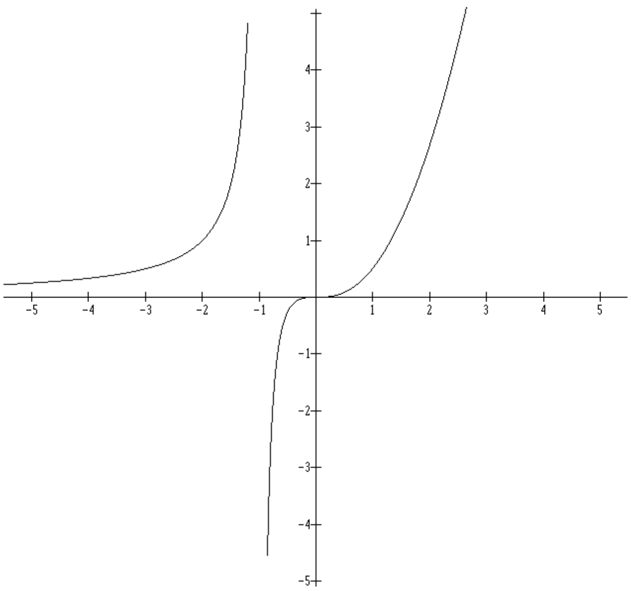
i) Find $\arg(z-1)$ in terms of θ. [1]

ii) Hence find $\arg(z^2-3z+2)$ in terms of θ. [2]

e) Find the complex fifth root of $-i$, in mod/arg form, and show these roots on an Argand diagram. [3]

Question Three: (15 Marks) *Start a new sheet of paper.*

a) The diagram shows the graph of $y=F(x)$. Draw neat sketches of (each should take about one third of a page):



i) $y=\dfrac{1}{F(x)}$ [2]

ii) $y=F(x)-|F(x)|$ [2]

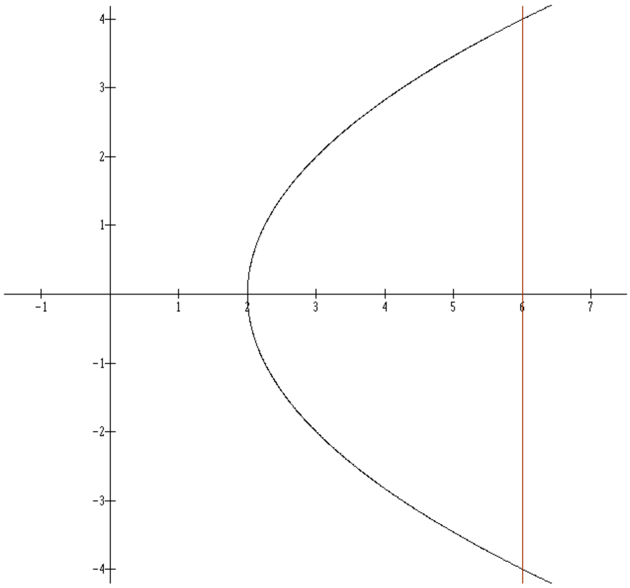
iii) $y=x.F(x)$ [2]

iv) $y=e^{F(x)}$ [2]

v) $y=\sqrt{F(x)}$ [2]

(Question 3 continued over)

b) The diagram shows the region bounded by the curve $y^2 = 4(x-2)$ and the line $x = 6$. Use the method of cylindrical shells to find the volume of the solid formed by rotating the given region about the $y-axis$. [5]



<u>Question Four:</u> (15 Marks) *Start a new sheet of paper.*

a)

i) Derive the tangent to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$. [2]

ii) If $P(a\cos\theta, b\sin\theta)$ is on the ellipse in the first quadrant, and the tangent at P meets the $x-axis$ and the $y-axis$ at X and Y respectively, find the coordinates of X and Y. [2]

iii) For the triangle thus formed by OXY, find the minimum area of this triangle, and the coordinates of P (in terms of a and b) for this case. [5]

b)

i) Given $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ where n is a positive integer and $n \geq 2$,

show that $I_n = \dfrac{1}{n-1} - I_{n-2}$. [4]

ii) Hence evaluate $I_5 = \int_0^{\frac{\pi}{4}} \tan^5 x dx$. [2]

Question Five: (15 Marks) *Start a new sheet of paper.*

a) Factorise $Q(x) = x^6 - 3x^2 + 2$ over the complex number field, given that it has two double roots. [3]

b) The equation $x^3 + px + 1 = 0$ has three real non-zero roots α, β and δ.

i) Find the values of $\alpha^2 + \beta^2 + \delta^2$ and $\alpha^4 + \beta^4 + \delta^4$ in terms of p, and show that p must be strictly negative. [4]

ii) Find the monic equation, with co-efficients in terms of p, whose roots are $\frac{\alpha}{\beta\delta}, \frac{\beta}{\alpha\delta}, \frac{\delta}{\alpha\beta}$. [4]

c) Let z_1, z_2 and z_3 be three complex numbers represented by the points Z_1, Z_2 and Z_3 respectively on the Argand diagram, where $z_1 \times z_3 = (z_2)^2$. Show that OZ_2 bisects $\angle Z_1OZ_3$. [4]

Question Six: (15 Marks) *Start a new sheet of paper.*

a) PQRS is a cyclic quadrilateral. The bisector of $\angle$ PQS cuts the segment PR at X and the circle at M, and RM cuts the segment QS at Y.

i) Draw a neat diagram showing the above information. [1]

ii) Prove XQRY is a cyclic quadrilateral. [3]

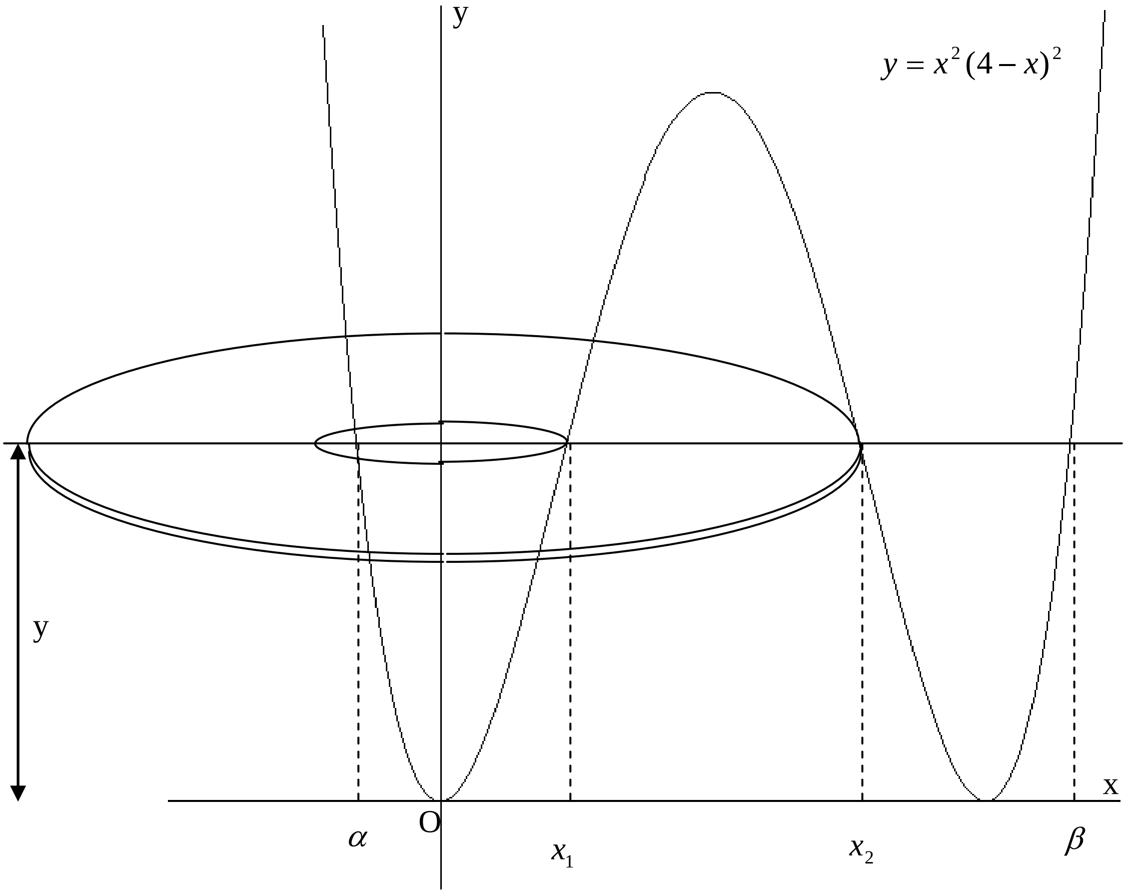
iii) Prove XY is parallel to PS. [3]

b) Find the limiting sum of the series $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + ... + \frac{n}{5^n} + ...$ [3]

c) From DeMoivre's Theorem, we know $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. Use this to solve the equation $16x^4 - 16x^2 + 1 = 0$, and deduce the exact values of $\cos\frac{\pi}{12}$ and $\cos\frac{5\pi}{12}$. [5]

Question Seven: (15 Marks) *Start a new sheet of paper.*

a)



The region between $x = 0$ and $x = 4$ is rotated about the $y-axis$. The volume of the solid formed is found by taking slices perpendicular the the $y-axis$. The typical slice shown in the diagram is at a height y above the $x-axis$.

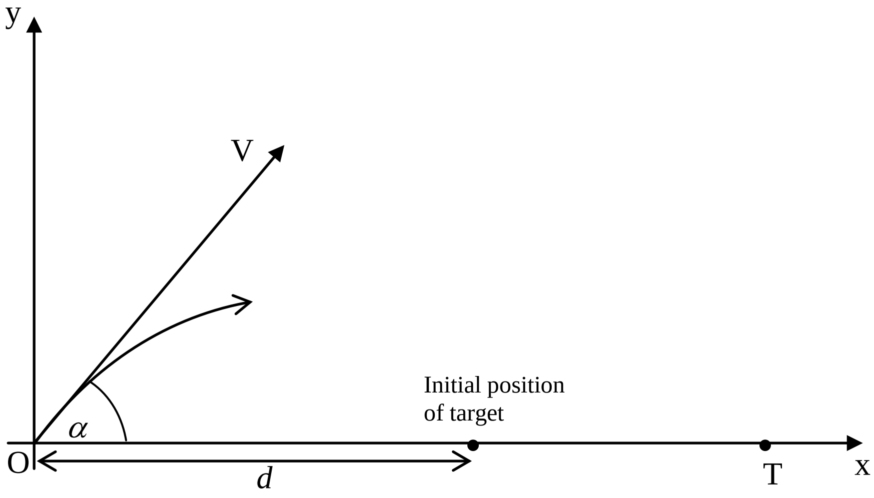
i) Deduce that α, x_1, x_2 and β, as shown in the diagram, are the roots of $x^4 - 8x^3 + 16x^2 - y = 0$. [1]

ii) Use the symmetry in the graph to explain why $\frac{x_1 + x_2}{2} = 2$ and $\frac{\alpha + \beta}{2} = 2$. Hence, by considering the co-efficients of the equation in (i), show that $\alpha\beta = -x_1 x_2$, and deduce that $x_1 x_2 = \sqrt{y}$ and that $x_2 - x_1 = 2\sqrt{4 - \sqrt{y}}$. [5]

(Question 8(a) continued over)

iii) Show that the volume of the solid of revolution is given by $V = 8\pi \int_0^{16} \sqrt{4-\sqrt{y}}\,dy$. Use the substitution $y = (4-u)^2$ to evaluate this integral and find the exact volume. [4]

b)



A projectile, of initial speed V m/s, is fired at an angle α from the origin O towards a target T, which is moving away from O along the $x-axis$.

You may assume that the projectile's trajectory is defined by the equations:

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$, where x and y are the horizontal and vertical displacements of the projectile in meters at time t seconds after firing, and where g is the acceleration due to gravity.

i) Show that the projectile is above the $x-axis$ for a total of $\frac{2V\sin\alpha}{g}$ seconds. [1]

ii) Show that the horizontal range of the projectile is $\frac{2V^2\sin\alpha\cos\alpha}{g}$ meters. [1]

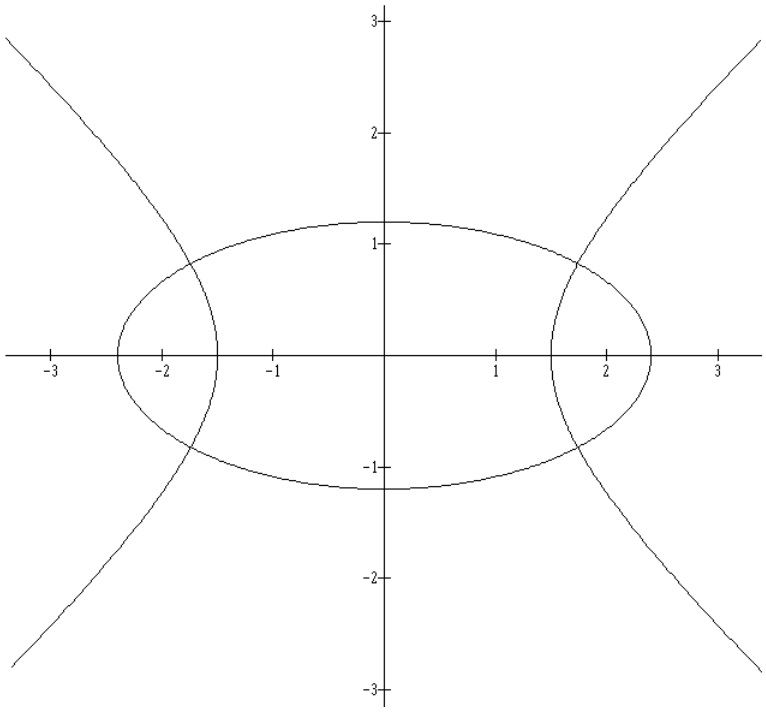
iii) At the instant the projectile is fired, the target T is d meters from O and is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α. Show that $u = V\cos\alpha - \frac{gd}{2V\sin\alpha}$. [3]

Question Eight: (15 Marks) *Start a new sheet of paper.*

a) Find the volume of the solid generated by rotating the region common to the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 8x$ about their common chord. [8]

b) Hyperbola H has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e. Ellipse E has equation $\frac{x^2}{(a^2 + b^2)} + \frac{y^2}{b^2} = 1$. See diagram below.



i) Show that ellipse E has eccentricity $\frac{1}{e}$. [1]

ii) If H and E intersect at P in the first quadrant, show that the acute angle α between the tangents to H and E at P is given by $\tan\alpha = \sqrt{2}\left(e + \frac{1}{e}\right)$. [6]

1) a) $\int \cos^3 x \, dx$

$= \int \cos x (1 - \sin^2 x) \, dx$

$= \int \cos x - \sin^2 x \cos x \, dx$

$= \sin x - \frac{1}{3}\sin^3 x + c$ //

b) $\int \frac{dx}{\sqrt{x^2+2x+5}}$

$= \int \frac{dx}{\sqrt{x^2+2x+1+4}}$

$= \int \frac{dx}{\sqrt{(x+1)^2+2^2}}$

$= \ln\left((x+1)^2 + \sqrt{(x+1)^2+4}\right) + c$ //

d) i) $\frac{1}{(x^2+1)(x+1)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{x+1}$

$\Rightarrow \quad 1 = (x+1)(Ax+B) + C(x^2+1)$

$x=-1$ gives $1 = 2C$ ie $C = \frac{1}{2}$

$x=0$ gives $1 = B + C$ ie $B = \frac{1}{2}$

$x=1$ gives $1 = 2(A+B) + 2C$ ie $A = \frac{-1}{2}$

$\therefore \frac{1}{(x^2+1)(x+1)} = \frac{1-x}{2(x^2+1)} + \frac{1}{2(x+1)}$ //

ii) $\int_0^1 \frac{1}{(x^2+1)(x+1)} \, dx$

$= \frac{1}{2}\int_0^1 \frac{1-x}{(x^2+1)} + \frac{1}{x+1} \, dx$

$= \frac{1}{2}\int_0^1 \frac{1}{x^2+1} + \frac{-x}{x^2+1} + \frac{1}{x+1} \, dx$

$= \frac{1}{2}\left[\tan^{-1}x - \frac{1}{2}\ln(x^2+1) + \ln(x+1)\right]_0^1$

$= \frac{1}{2}\left[\tan^{-1}1 - \frac{1}{2}\ln 2 + \ln 2 - (\tan^{-1}0 - \frac{1}{2}\ln 1 + \ln 1)\right]$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\ln 2 - 0\right)$

$= \frac{\pi + 2\ln 2}{8}$ //

Marking Guidelines

- a) ① appropriate substitution
- ① correct soln inc. "+c"
- b) ① correct manipulation of root to a standard form
- ① correct use of standard integral.
- d) i) ① forms correct equivalent equations
- ① correct values of a, b, c
- ii) ① correct integrations
- ① answer correct

Comments

- a) · well done
- b) · some students had difficulty once the root was in a standard form
- d) i) · well done
- ii) · well done

Q 1) d) $\int_1^2 x \ln x \, dx$ $\quad u = \ln x \quad dv = x$ $\quad du = \frac{1}{x} \quad v = \frac{1}{2}x^2$

$\therefore = \left[\frac{1}{2}x^2 \ln x\right]_1^2 - \int_1^2 \frac{1}{2}x^2 \cdot \frac{1}{x} \, dx$

$= \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^2$

$= \left(2\ln 2 - 1 - (0 - \frac{1}{4})\right)$

$= 2\ln 2 - \frac{3}{4}$

$= \frac{4\ln 2 - 3}{4}$ //

e) $\int_0^{\frac{1}{2}} \frac{\sqrt{x}\,dx}{(1-x)^{3/2}}$

$x = \sin^2\theta$

$dx = 2\sin\theta . \cos\theta \, d\theta$

$x = \frac{1}{2} \quad \sin^2\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{4}$

$x = 0 \quad \sin^2\theta = 0 \Rightarrow \theta = 0$

$\therefore = \int_0^{\frac{\pi}{4}} \frac{\sqrt{\sin^2\theta} . 2\sin\theta\cos\theta\,d\theta}{(1-\sin^2\theta)^{3/2}}$

$= 2\int_0^{\frac{\pi}{4}} \frac{\sin^2\theta\cos\theta}{\cos^3\theta} \, d\theta$

$= 2\int_0^{\frac{\pi}{4}} \tan^2\theta \, d\theta$

$= 2\int_0^{\frac{\pi}{4}} (\sec^2\theta - 1) \, d\theta$

$= 2\left[\tan\theta - \theta\right]_0^{\frac{\pi}{4}}$

$= 2\left(1 - \frac{\pi}{4} - (0)\right)$

$= 2 - \frac{\pi}{2}$ //

Marking Guidelines

- d) ① correct by parts u, v
- ① correct definite integral method
- ① correct answer
- e) ① correct change of limits
- ① resolves to $\tan^2$
- ① correct $\tan^2$ integration
- ① correct answer

Comments

- d) · well done
- e) · well done apart from careless errors such as omitting 2.

	Marking Guidelines	Comments

Q2) a) $z = \sqrt{3} + \frac{1+i}{1-i} \times \frac{(1+i)}{(1+i)}$

$= \sqrt{3} + \frac{2i}{2}$

$= \sqrt{3} + i$

$\therefore$ (i) $\text{Im}(z) = 1$ — ① — • Some wished to include the "i" in $\text{Im}(z)$.

(ii) $\bar{z} = \sqrt{3} - i$ — ①

(iii) $|z| = \sqrt{a^2+b^2}$ $\quad \arg(z) = \tan^{-1}\frac{1}{\sqrt{3}}$ — • some did mod/arg of $\bar{z}$.

$= \sqrt{3+1}$ $\quad = \frac{\pi}{6}$ — ① mod

$= 2$ — ① angle

$\therefore z = 2\,\text{cis}\,\frac{\pi}{6}$

b) $z^2 = 3-4i$

ie $(a+ib)^2 = 3-4i$ $\quad$ a, b real

$a^2-b^2+2abi = 3-4i$

$\therefore a^2-b^2 = 3$ — ① $\quad 2ab = -4$

$b = \frac{-2}{a}$ — ② in ①

$\therefore a^2 - \left(\frac{-2}{a}\right)^2 = 3$

$a^2 - \frac{4}{a^2} = 3$ — ① appropriate method for getting $\sqrt{\ }$

$\therefore a^4 - 3a^2 - 4 = 0$

$(a^2-4)(a^2+1) = 0$

so $a^2 = 4$ or $a^2 = -1$ (reject as a is real) — • remember to write down solutions, not just state a and b!

$\therefore a = \pm 2 \quad b = \mp 1$ — ① correct solutions for z_1, z_2

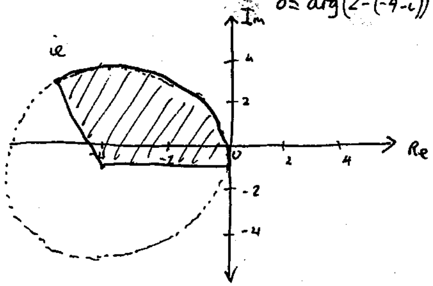
$\therefore z_1 = 2-i \quad z_2 = -2+i$ //

c) $z+4+i$ $\Rightarrow |z-(-4-i)| \le 4$, a circle centre $(-4-i)$ — ① correct $|\ |$ interpret.

$= z-(-4-i)$

$0 \le \arg(z-(-4-i)) \le \frac{2\pi}{3}$ $\Rightarrow$ sector. — ① correct arg interpret.

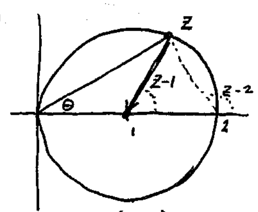
ie — ① correct diagram. — • some placed the center not at $(-4,-1)$.



//

	Marking Guidelines	Comments

d)



• those who drew a diagram did quite w[ell]. Those that did not, [did] very poorly on this question!

i) $\arg(z-1) = 2\theta$ — ①

ii) $\arg(z^2-3z+2) = \arg[(z-2)(z-1)]$

$= \arg(z-2) + \arg(z-1)$

$\arg(z-2)$ is external angle of Δ, so — ① correct $\arg(z-2)$ — • many tried to use π o[n] the straight line - unsuccessfully.

$\arg(z-2) = \frac{\pi}{2} + \theta$

$\therefore \arg(z^2-3z+2) = \arg(z-2) + \arg(z-1)$

$= \frac{\pi}{2} + \theta + 2\theta$ — ① correct answer

$= 3\theta + \frac{\pi}{2}$ //

e) $-i = \text{cis}\left(\frac{-\pi}{2}\right)$ — • many had wrong cis θ [for] $-i$!

let $z = r\,\text{cis}\,\theta$

$\therefore z^5 = r^5\,\text{cis}\,5\theta$ (by DeMoivres Theorem) — • many did not make clear connection in the working, between DMT and eqn.

$\therefore r^5\,\text{cis}\,5\theta = \text{cis}\left(-\frac{\pi}{2}\right)$ — ① correct use of DeMoivre and general solution.

$\therefore r = 1, \quad 5\theta = 2k\pi - \frac{\pi}{2}$

$\therefore \theta = \frac{\pi}{10}(4k-1) \quad k = 0,1,2,3,4.$ — • general soln poorly don[e]

$\therefore z_0 = \text{cis}\left(\frac{-\pi}{10}\right) \left(= \text{cis}\,\frac{19\pi}{10}\right)$

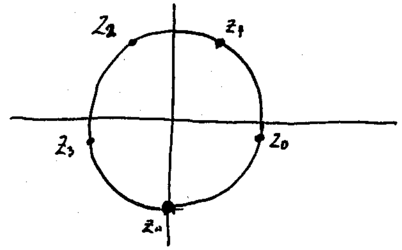
$z_1 = \text{cis}\left(\frac{3\pi}{10}\right)$ — ① correct roots listed

$z_2 = \text{cis}\left(\frac{7\pi}{10}\right)$

$z_3 = \text{cis}\left(\frac{11\pi}{10}\right)$

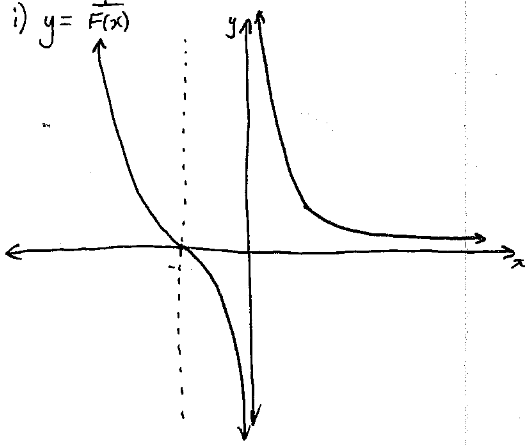
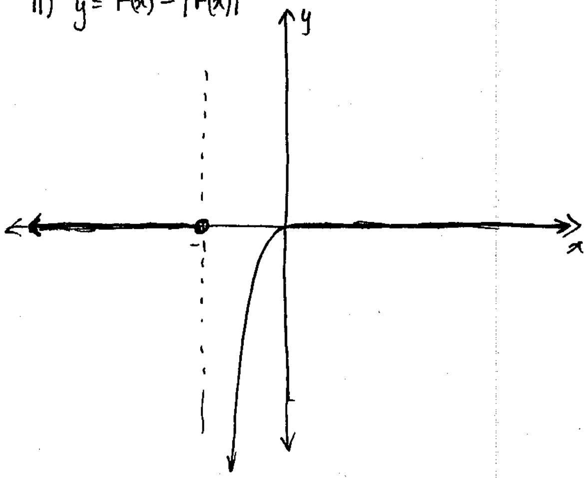
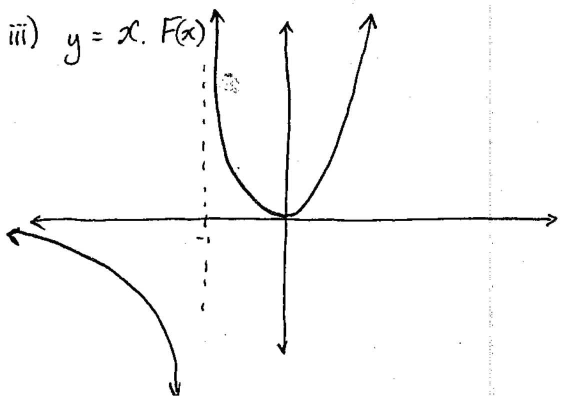
$z_4 = \text{cis}\left(\frac{15\pi}{10}\right)$

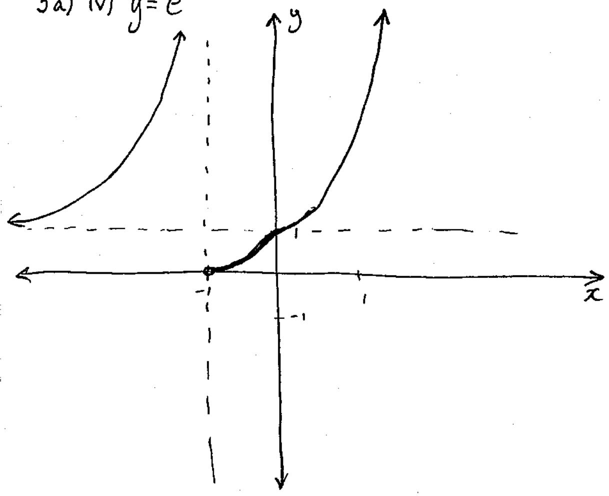
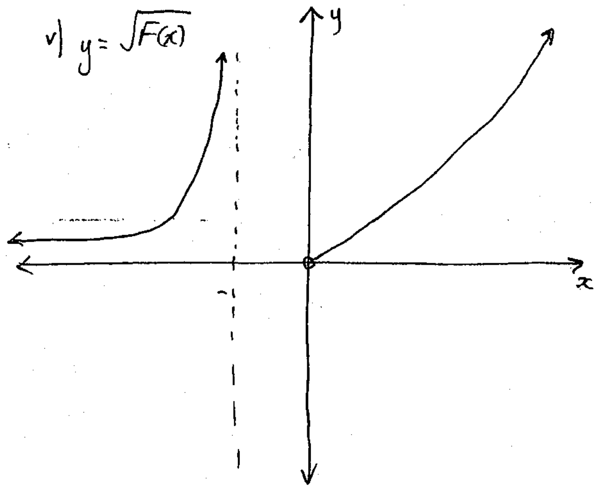
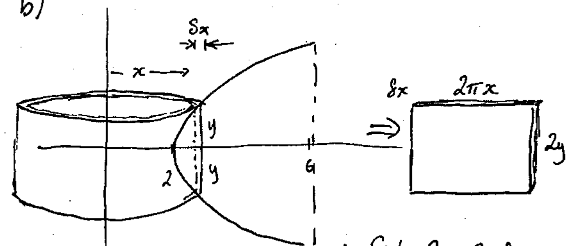
$= \text{cis}\left(\frac{3\pi}{2}\right) \ (= -i)$



① diagram correct — • many incorrect due [to] not graphing $-i$ correctly!

//

	Marking Guidelines	Comments
3) a) i) $y = \frac{1}{F(x)}$ 	① correct reversal of zero/asymptote ① correct shape	· well done
ii) $y = F(x) - \lvert F(x) \rvert$ 	① correct exclusion of asymptote ① zero in correct areas	· care needed to show where curve and x-axes co-incide.
iii) $y = x.\ F(x)$ 	① correct asymptote + zeros ① correct shape/regions	

	Marking Guidelines	Comments
3a) iv) $y = e^{F(x)}$ 	① correct behaviour about $x=0$ ① correct behaviour at $x=-1$	
v) $y = \sqrt{F(x)}$ 	① correct region where $\sqrt{\ }$ doesn't exist ① correct shape.	· care needed when showing where $\sqrt{\ }$ doesn't exist.
b)  ∴ $\delta V = 2\pi x 2y\, \delta x$	① set up problem with correct δV. (must have some sort of relevant diagram!)	

3 b) (cont)

$\therefore V = \lim_{\delta x \to 0} \sum_{x=2}^{6} 4\pi y x \, \delta x$

$= 4\pi \int_2^6 xy \, dx$

$= 4\pi \int_2^6 2x\sqrt{x-2} \, dx$

using $u = x-2$: $du = dx$ and $x = u+2$

$x=6, u=4$: $x=2, u=0$

$\therefore V = 8\pi \int_0^4 (u+2)\sqrt{u} \, du$

$= 8\pi \int_0^4 u^{\frac{3}{2}} + 2u^{\frac{1}{2}} \, du$

$= 8\pi \left[\frac{2}{5} u^{\frac{5}{2}} + \frac{4}{3} u^{\frac{3}{2}} \right]_0^4$

$= 8\pi \left[\left(\frac{2}{5} \cdot 4^{\frac{5}{2}} + \frac{4}{3} \cdot 4^{\frac{3}{2}} \right) - 0 \right]$

$= \frac{2816\pi}{15}$ cubic units. //

4) a) i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ (by implicit different.)

$\therefore \frac{dy}{dx} = \frac{-2x}{a^2} \cdot \frac{b^2}{2y}$

$= -\frac{b^2 x}{a^2 y}$

at $P(a\cos\theta, b\sin\theta)$:

$\frac{dy}{dx} = -\frac{b^2 \cdot a\cos\theta}{a^2 \cdot b\sin\theta}$

$= -\frac{b\cos\theta}{a\sin\theta}$

$\therefore$ tangent is:

$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

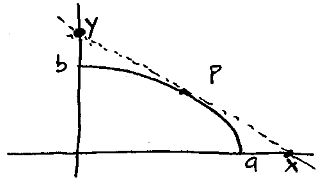
$ay\cos\theta - ab\sin^2\theta = -xb\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab\cos^2\theta + ab\sin^2\theta$

$\div ab$ $\quad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \sin^2\theta + \cos^2\theta$

$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ //

ii)



Marking Guidelines

- ① resolving integral to x's.
- ① correct substitutions (inc limits).
- ① correct integration.
- ① correct answer
- ① correct $\frac{dy}{dx}$ at P. (inc. differentiation shown)
- ① correct algebra to tangent eqn.

Comments

- many students were unable to complete the integration.
- many still unable to reproduce this basic proceedure!
- many answers not reduced to the standard form.

4) a) (ii) (cont)

at X: $y=0 \Rightarrow \frac{X\cos\theta}{a} = 1$

$\therefore X = \frac{a}{\cos\theta}$

$\therefore X$ is $\left(\frac{a}{\cos\theta}, 0\right)$

at Y: $x=0 \Rightarrow \frac{Y\sin\theta}{b} = 1$

$\therefore Y = \frac{b}{\sin\theta}$

$\therefore Y$ is $\left(0, \frac{b}{\sin\theta}\right)$ //

iii) $A_{oxy} = \frac{1}{2} \cdot OX \cdot OY$

$= \frac{1}{2} \cdot \frac{a}{\cos\theta} \cdot \frac{b}{\sin\theta}$

$= \frac{ab}{2\sin\theta\cos\theta}$

$= \frac{ab}{\sin 2\theta}$.

$\therefore \frac{dA}{d\theta} = -1 \cdot ab(\sin 2\theta)^{-2} \cdot 2\cos 2\theta$.

$= \frac{-2ab\cos 2\theta}{\sin^2 2\theta}$

for $\frac{dA}{d\theta} = 0 \Rightarrow -2ab\cos 2\theta = 0$

or $\cos 2\theta = 0$

$2\theta = \frac{\pi}{2}$ (1st Quad)

ie $\theta = \frac{\pi}{4}$

test for min: $\frac{d^2A}{d\theta^2} = \frac{+4ab\sin 2\theta}{\sin^2 2\theta} + \frac{-2ab\cos 2\theta \cdot -2 \cdot 2\cos 2\theta}{\sin^3 2\theta}$

$= \frac{+4ab\sin^2 2\theta + 8ab\cos^2 2\theta}{\sin^3 2\theta}$

when $\theta = \frac{\pi}{4}$: $\frac{d^2A}{d\theta^2} = \frac{+4ab\sin^2(\frac{\pi}{2}) + 8ab\cos^2(\frac{\pi}{2})}{\sin^3(\frac{\pi}{2})}$

$= 4ab$

$> 0 \Rightarrow$ rel. min.

$\therefore \text{Area}_{min} = \frac{ab}{\sin(\frac{\pi}{2})}$

$= ab$ sq. units

when P is $\left(a\cos\frac{\pi}{4}, b\sin\frac{\pi}{4}\right)$

or $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ //

Marking Guidelines

- ① finding X
- ① finding Y.
- ① expression for area
- ① expression for $\frac{dA}{d\theta}$
- ① correct value of $\theta = \frac{\pi}{4}$
- ① showing rel. min.
- ① co-ords of P.

Comments

- generally well done
- no-one showed the area a minimum! This is basic min/max proceed
- many missed this part out.

b) i) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x . \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x \, dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x \, dx$

$= \left[\frac{1}{n-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{n-2}$

$= \left(\frac{1}{n-1} \tan^{n-1} \frac{\pi}{4} - \frac{1}{n-1} \cdot 0\right) - I_{n-2}$

$= \frac{1}{n-1} - I_{n-2}$ //

ii) from $I_n = \frac{1}{n-1} - I_{n-2}$

then $I_5 = \frac{1}{4} - I_3$

$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$

now $I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \left[\ln(\cos x)\right]_0^{\frac{\pi}{4}}$

$= \ln \frac{1}{\sqrt{2}} - \ln 1$

$= \ln 2^{-\frac{1}{2}}$

$= -\frac{1}{2} \ln 2$

$\therefore I_5 = \frac{1}{4} - \left(\frac{1}{2} - -\frac{1}{2} \ln 2\right)$

$= \frac{1}{2} + \frac{1}{2} \ln 2$

$= \frac{1}{2}(1 + \ln 2)$ //

5) a) $Q(x) = x^6 - 3x^2 + 2$

$Q'(x) = 6x^5 - 6x$

$= 6x(x^4 - 1)$

roots of $Q'(x)$ are $0, \pm 1, \pm i$

$Q(0) \neq 0 \quad Q(1) = 0 \quad Q(i) \neq 0$

$Q(-1) = 0 \quad Q(-i) \neq 0$

$\therefore \pm 1$ are the double roots.

Marking Guidelines

① correct $\tan^n x$ method

① correct integrals

① correct substitutions

① recognises I_{n-2}.

① correct evaluation of I_1

① correct use of reduction formula for answer.

① double roots clearly identified (with reasons).

Comments

- most could start this proceedure.
- many tried "by parts" at this point! Why?
- many did not use reduction formula properly - tried to find I_3.
- Evaluate I_1 or I_0 in a reduction formula!
- many not able to resolve $\ln \frac{1}{\sqrt{2}}$ to its $\ln 2$ equivalent.

5) a) (cont).

$\therefore Q(x) = (x-1)^2(x+1)^2 R(x)$

$= (x^2 - 2x + 1)(x^2 + 2x + 1) R(x)$

$= (x^4 - 2x^2 + 1) . R(x)$

$\therefore R(x) = \frac{Q(x)}{(x^4 - 2x^2 + 1)}$

$$\begin{array}{r|l} & x^2 + 2 \\ x^4 - 2x^2 + 1 & x^6 \quad\quad - 3x^2 + 2 \\ & x^6 - 2x^4 + x^2 \\ \hline & 2x^4 - 4x^2 + 2 \\ & 2x^4 - 4x^2 + 2 \\ \hline \end{array}$$

$\therefore Q(x) = (x-1)^2(x+1)^2(x^2+2)$

$= (x-1)^2(x+1)^2(x + \sqrt{2} i)(x - \sqrt{2} i)$ //

b) i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

now $\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = p$

$\alpha\beta\gamma = +1$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 0^2 - 2(p)$

$= -2p$

since α, β and γ are non-zero and real

$-2p > 0$, so

$p < 0$ //

for $\alpha^4 + \beta^4 + \gamma^4$: $\quad x^3 = -px - 1$

so $\quad x^4 = -px^2 - x$

ie $\alpha^4 = -p\alpha^2 - \alpha$

$\beta^4 = -p\beta^2 - \beta$

$\gamma^4 = -p\gamma^2 - \gamma$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = -p(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma)$

$= -p(-2p) - 0$

$= 2p^2$

Marking Guidelines

① correct $R(x)$ (any method - must show working)

① factored over complex field

① correct value of $\alpha^2 + \beta^2 + \gamma^2$

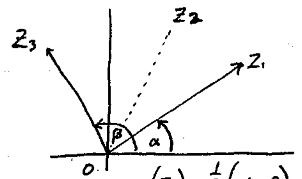
① reasoning for $p < 0$

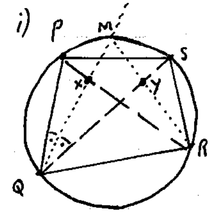
① method for x^4.

① correct value of $\alpha^4 + \beta^4 + \gamma^4$

Comments

- well done
- well done except for careless errors

	Marking Guidelines	Comments
5) b) (ii)		
for roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\alpha\gamma}, \frac{\gamma}{\alpha\beta}$, noting $\alpha\beta\gamma=-1$ gives $\frac{\alpha}{\alpha\beta\gamma} = \frac{\alpha^2}{\alpha\beta\gamma} = -\alpha^2$ ie $x = -\alpha^2$ so $\alpha = \sqrt{-x}$	① use of $\alpha\beta\gamma=-1$ ① using α to form eqn.	
in original equation: $(\sqrt{-x})^3 + p\sqrt{-x} + 1 = 0$ $\sqrt{-x}\,(p - x) = -1$ squaring: $-x(p-x)^2 = 1$ $-x(p^2 - 2px + x^2) = 1$ $-p^2x + 2px^2 - x^3 = 1$ so $0 = x^3 - 2px^2 + p^2x + 1$ is required eqn. //	① working, including squaring to resolve $\sqrt{\ }$ ① final eqn.	
c) $z_1 . z_3 = (z_2)^2$ ie $\arg(z_1) + \arg(z_3) = 2\arg(z_2)$ ie $\arg(z_2) = \frac{1}{2}(\arg(z_1) + \arg(z_3))$	① correct arg relationships ① expression for $\arg z_2$	• not well done - many students have difficulty connecting complex numbers with geometry.
 $\arg(z_2) = \frac{1}{2}(\alpha+\beta)$	① diagram showing relationships is correct	
ie $\arg(z_2) - \arg(z_1) = \arg(z_3) - \arg(z_2)$ ie $\angle Z_2OZ_1 = \angle Z_3OZ_2$ $\Rightarrow OZ_2$ bisects $\angle Z_3OZ_1$ //	① correct interpretation linking α, β with args	

	Marking Guidelines	Comments
6) a) i)  //	① diagram correct.	• poorly done • diagrams were generally not neat
ii) Prove $XQRY$ a cyclic quadrilateral: $\angle PQM = \angle PRM$ (angles at circumference standing on arc PM) $\angle PQM = \angle SQM$ (given QM bisects $\angle PQS$) $\therefore \angle SQM = \angle PRM$ (both $= \angle PQM$) $\therefore$ 'arc XY' subtends equal angles at Q and R ie $\angle XQY (\equiv \angle SQM) = \angle XRY (\equiv \angle PRM)$ $\therefore XQRY$ is a cyclic quadrilateral. //	① linking $\angle Q$ to $\angle R$ ① using bisector ① recognising subtended angles on arc XY.	
iii) $\angle RQY = \angle RXY$ (angles at circumference standing on arc RY) $\angle RPS = \angle RQS$ (angles at circumference standing on arc RS) $\therefore \angle RXY = \angle RPS$ (both equal to $\angle RQY$ $(\equiv \angle RQS)$) $\therefore XY \parallel PS$ (corresponding angles are equal) //	① angles from both cyclic quads ① linking these angles ① conclusion (with reason)	
b) $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots$ $= \frac{1}{5} + (\frac{1}{5^2} + \frac{1}{5^2}) + (\frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{5^3}) + \ldots$ $= (\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots) + (\frac{1}{5^2} + \frac{1}{5^3} + \ldots) + (\frac{1}{5^3} + \frac{1}{5^4} + \ldots)$ each is a GP with $r = \frac{1}{5}$, and differing a's.	① rearranging pattern	• most had trouble seeing any pattern
ie: (with $S_\infty = \frac{a}{1-r}$): $= \frac{\frac{1}{5}}{1-\frac{1}{5}} + \frac{\frac{1}{5^2}}{1-\frac{1}{5}} + \frac{\frac{1}{5^3}}{1-\frac{1}{5}} + \ldots$ $= \frac{\frac{1}{5}}{4/5} + \frac{\frac{1}{5^2}}{4/5} + \frac{\frac{1}{5^3}}{4/5} + \ldots$ $= \frac{1}{4} + \frac{1}{20} + \frac{1}{100} + \ldots$ another GP with $a = \frac{1}{4}$, $r = \frac{1}{5}$	① 1st ~~GP~~ groups.	

6). b) (cont)

$\therefore S_\infty = \dfrac{\frac{1}{4}}{4/5}$

$= \frac{5}{16}$. //

6). c) Let $x = \cos\theta$

$\therefore 16\cos^4\theta - 16\cos^2\theta + 1 = 0$

ie $8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$

or $\cos 4\theta = \frac{1}{2}$

$\therefore 4\theta = 2n\pi \pm \cos^{-1}(\frac{1}{2})$

$= 2n\pi \pm \frac{\pi}{3}$

$= \frac{\pi(6n\pm1)}{3}$

$\therefore \theta = \left(\frac{6n\pm1}{12}\right)\pi \quad n = 0, \pm1, \pm2$

$\therefore$ for $0 \le \theta \le 2\pi$: $\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

for $x = \cos\theta$; unique values are

$x_1 = \cos\frac{\pi}{12} \quad (= \cos\frac{23\pi}{12})$

$x_2 = \cos\frac{5\pi}{12} \quad (= \cos\frac{19\pi}{12})$

$x_3 = \cos\frac{7\pi}{12} \quad (= \cos\frac{17\pi}{12})$

$x_4 = \cos\frac{11\pi}{12} \quad (= \cos\frac{13\pi}{12})$

we also note $\cos\frac{\pi}{12} = -\cos\frac{11\pi}{12} \quad (x_1 = -x_4)$

and $\cos\frac{5\pi}{12} = -\cos\frac{7\pi}{12} \quad (x_2 = -x_3)$

Similarly, by using $u = x^2$, we get

$16u^2 - 16u + 1 = 0$

ie $u = \frac{16 \pm \sqrt{192}}{32}$

$= \frac{16 \pm 8\sqrt{3}}{32}$

$= \frac{1}{4}(2 \pm \sqrt{3})$

$\therefore x^2 = \frac{1}{4}(2 \pm \sqrt{3})$

so $x = \pm\frac{1}{2}\sqrt{2 \pm \sqrt{3}}$

These must correspond to the x_1 to x_4 above, so $\cos\frac{\pi}{12}$ is the largest positive, ~~then~~:

$\cos\frac{\pi}{12} = \frac{1}{2}\sqrt{2+\sqrt{3}}$

and $\cos\frac{5\pi}{12} = \frac{1}{2}\sqrt{2-\sqrt{3}}$ (the other +ve) //

Marking Guidelines

- ① 2nd GP with correct soln.
- ① values for θ - generator or other method.
- ① unique values for roots
- ① roots in terms of surds.
- ① $\cos\frac{\pi}{12}$ correct
- ① $\cos\frac{5\pi}{12}$ correct (both with reason).

Comments

- this is a fairly standard question - poorly done (no matter which method was used).
- many had difficulties resolving the unique roots.
- many could not do this part.
- many could not indicate why they made their choice.

7) (a)

i) α, x_1, x_2 and β satisfy $y = x^2(4-x)^2$ for the given y value. Considering y constant;

$\therefore x^2(4-x)^2 - y = 0$

$x^2(16 - 8x + x^2) - y = 0$

or $x^4 - 8x^3 + 16x^2 - y = 0$

ii) from the graph, $x = 2$ is an axis of symmetry, so 2 is the midpoint between x_1 and x_2, α and β.

$\therefore \frac{x_1+x_2}{2} = 2 \qquad \frac{\alpha+\beta}{2} = 2$

ie $x_1 + x_2 = 4 \qquad \alpha + \beta = 4$

$\therefore x_1 + x_2 + \alpha + \beta = 8$ (from co-eff, as well as above)

by 3 roots at a time:

$x_1x_2\alpha + x_1x_2\beta + x_1\alpha\beta + x_2\alpha\beta = 0$

$(\alpha+\beta)x_1x_2 + (x_1+x_2)\alpha\beta = 0$

but $\alpha+\beta = 4$, $x_1 + x_2 = 4$ gives

$4x_1x_2 + 4\alpha\beta = 0$

ie $\alpha\beta = -x_1x_2$ as reqd. //

from 4 roots together:

$\alpha\beta x_1x_2 = -y$

ie $x_1x_2(-x_1x_2) = -y$

gives $(x_1x_2)^2 = y$

or $x_1x_2 = \sqrt{y}$

for $x_2 - x_1$, we note

$(x_2 - x_1)^2 = (x_2+x_1)^2 - 4x_1x_2$

$= 4^2 - 4\sqrt{y}$

$= 16 - 4\sqrt{y}$

as $x_2 > x_1$,

$x_2 - x_1 = +\sqrt{16 - 4\sqrt{y}}$

$= 2\sqrt{4 - \sqrt{y}}$ //

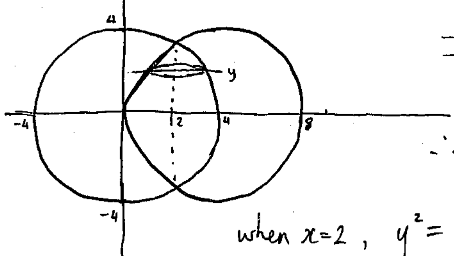
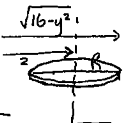
Marking Guidelines

- ① correct deduction + re-arrangement of equation.
- ① correct explanation
- ~~① correct roots x3~~
- ① correct working to result.
- ① correct working to result
- ① expression for $(x_2 - x_1)^2$
- ① correct working to result.

Comments

- well done
- well done
- These steps proved difficult for many students

Solution	Marking Guidelines	Comments
7)(a) iii) The slice has volume δV, given by		
$\delta V = \pi(x_2^2 - x_1^2)\delta y$		
$= \pi(x_2 - x_1)(x_2 + x_1)\delta y$		• most students could establish V as required.
$= 8\pi\sqrt{4-\sqrt{y}}\,\delta y$ (using results from (ii))	① correct δV	
Hence $V = \lim_{\delta y \to 0} \sum_{y=0}^{16} 8\pi\sqrt{4-\sqrt{y}}\,\delta y$		
$= 8\pi \int_0^{16} \sqrt{4-\sqrt{y}}\, dy$	① correct working to show V.	
using $y = (4-u)^2$, $dy = -2(4-u)\,du$; $y=0,\ u=4$; $y=16,\ u=0$		
$4-\sqrt{y} = 4-(4-u) = u$	① correct limits for sub.	
$\therefore V = 8\pi \int_4^0 \sqrt{u}\,.(-2(4-u))\,du$		
$= 16\pi \int_4^0 u^{3/2} - 4u^{1/2}\,du$		
$= 16\pi\left[\frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\right]_4^0$		
$= 16\pi\left[0 - \left(\frac{2}{5}4^{5/2} - \frac{8}{3}4^{3/2}\right)\right]$		
$= 16\pi\left(-\frac{64}{5} + \frac{64}{3}\right)$		
$= \frac{2048\pi}{15}$ cubic units	① correct answer.	
b) i) from $y = Vt\sin\alpha - \frac{1}{2}gt^2$		
$y=0$: $0 = Vt\sin\alpha - \frac{1}{2}gt^2$		
$= t(V\sin\alpha - \frac{1}{2}gt)$	① working correct to result.	• mostly well done
ie $t=0$ or $0 = V\sin\alpha - \frac{1}{2}gt$		
$\frac{1}{2}gt = V\sin\alpha$		
so $t = \frac{2V\sin\alpha}{g}$ as reqd		

Solution	Marking Guidelines	Comments
7) b) ii) max range when $t = \frac{2V\sin\alpha}{g}$		
in $x = Vt\cos\alpha$	① correct subst.	
$= \frac{2V^2\sin\alpha\cos\alpha}{g}$ m		
iii) position of target T is given by $x = d + ut$		
to hit the target, $t = \frac{2V\sin\alpha}{g}$ and $x = \frac{2V^2\sin\alpha\cos\alpha}{g}$	① initial setup correct	
substituting in above; $\frac{2V^2\sin\alpha\cos\alpha}{g} = d + u\cdot\frac{2V\sin\alpha}{g}$		
ie $u\cdot\frac{2V\sin\alpha}{g} = \frac{2V^2\sin\alpha\cos\alpha - dg}{g}$	① substitutions correct	
$\therefore u = \frac{2V^2\sin\alpha\cos\alpha - gd}{2V\sin\alpha}$		
$= V\cos\alpha - \frac{gd}{2V\sin\alpha}$	① correct working to answer.	
8) a) $x^2+y^2=16$, center (0,0), radius 4		
$x^2-8x+y^2=0$; $x^2-8x+16+y^2=16$; center (4,0) radius 4.		
common chord is $x=2$ (by symmetry)	① correct circles (including diagram) and common chord	
 		
$\therefore R = \sqrt{16-y^2} - 2$	① correct R	• most did not draw a slice/shell diagram so got an incorrect radius expression
when $x=2$, $y^2=12$, $\therefore y = 2\sqrt{3}$	① correct y limit	• many used shells - very poorly! Shells works best when the axis of rotation is outside the region being rotated.
for slice at y: $SA = \pi(\sqrt{16-y^2} - 2)^2$	① correct expression for SA	
$= \pi(16 - y^2 - 4\sqrt{16-y^2} + 4)$		
$= \pi(20 - y^2 - 4\sqrt{16-y^2})$		

8) a) (cont)

$\therefore \delta V = \delta A . \delta y$

$= \pi\left(20 - y^2 - 4\sqrt{16-y^2}\right)\delta y$

$\therefore V = \lim_{\delta y \to 0} \sum_{y=-2\sqrt{3}}^{2\sqrt{3}} \pi\left(20-y^2-4\sqrt{16-y^2}\right)\delta y$

$= \int_{-2\sqrt{3}}^{2\sqrt{3}} \pi\left(20-y^2-4\sqrt{16-y^2}\right) dy$

$= 2\pi\int_0^{2\sqrt{3}} 20-y^2-4\sqrt{16-y^2}\, dy$ (from symmetry)

$= 2\pi\left[20y - \frac{1}{3}y^3\right]_0^{2\sqrt{3}} - 2\pi\int_0^{2\sqrt{3}} 4\sqrt{16-y^2}\, dy$

using $y = 4\sin\theta$ $\quad y=0 \Rightarrow \theta = 0$

$dy = 4\cos\theta\, d\theta$ $\quad y = 2\sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$\therefore$

$= 2\pi\left[40\sqrt{3} - \frac{1}{3}(2\sqrt{3})^3 - (0)\right] - 8\pi\int_0^{\frac{\pi}{3}} \sqrt{16-16\sin^2\theta}\,.4\cos\theta\, d\theta$

$= 2\pi\left(40\sqrt{3} - 8\sqrt{3}\right) - 128\pi\int_0^{\frac{\pi}{3}} \cos^2\theta\, d\theta$

$= 64\pi\sqrt{3} - 128\pi\int_0^{\frac{\pi}{3}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$

$= 64\pi\sqrt{3} - 64\pi\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$

$= 64\pi\left(\sqrt{3} - \left[\left(\frac{\pi}{3} + \frac{1}{2}.\frac{\sqrt{3}}{2}\right) - 0\right]\right)$

$= 64\pi\left(\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right)$ cu. units //

b) (i) for H: $b^2 = a^2(e^2-1)$

$e^2 - 1 = \frac{b^2}{a^2}$

so $e^2 = \frac{b^2}{a^2} + 1$

$= \frac{b^2+a^2}{a^2}$

For E; call its eccentricity ε

$\therefore b^2 = (a^2+b^2)(1-\varepsilon^2)$

ie $1 - \varepsilon^2 = \frac{b^2}{a^2+b^2}$

or $\varepsilon^2 = 1 - \frac{b^2}{a^2+b^2}$

Marking Guidelines

① correct expression for V

① working to substitution

① correct working in substitution

① correct answer

Comments

· those using shells set up the wrong radius expression with the wrong 'y' expression and usually the wrong limits!

· use a different symbol for the eccentricity of the ellipse!

8) b) (i) (cont)

$\therefore \varepsilon^2 = \frac{(a^2+b^2) - b^2}{a^2+b^2}$

$= \frac{a^2}{a^2+b^2}$

$= \frac{1}{e^2}$

$\therefore \varepsilon = \frac{1}{e}$

hence the ellipse E has eccentricity $\frac{1}{e}$ //

ii) For P in quadrant 1;

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ —① $\quad \frac{x^2}{(a^2+b^2)} + \frac{y^2}{b^2} = 1$

but from (i) $e^2 = \frac{a^2+b^2}{a^2}$

so $a^2 + b^2 = a^2e^2$

giving $\frac{x^2}{a^2e^2} + \frac{y^2}{b^2} = 1$ —②

①+② gives: $\frac{x^2}{a^2e^2} + \frac{x^2}{a^2} = 2$

$\frac{x^2}{a^2}\left(\frac{1}{e^2} + 1\right) = 2$

or $\frac{x^2}{a^2e^2}(1+e^2) = 2$

$e^2 \times$②: $\frac{x^2}{a^2} + \frac{e^2y^2}{b^2} = e^2$ —③

so ③−① gives: $\frac{e^2y^2}{b^2} + \frac{y^2}{b^2} = e^2 - 1$

$\frac{y^2}{b^2}(e^2+1) = e^2 - 1$

$\therefore$ at P $x^2 = \frac{2a^2e^2}{e^2+1}$ $\quad y^2 = \frac{b^2(e^2-1)}{e^2+1}$

so $x = ae\sqrt{\frac{2}{e^2+1}}$ so $y = b\sqrt{\frac{e^2-1}{e^2+1}}$

for tan α, we need the ~~angle~~

at P on H: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}$

at P: $\frac{dy}{dx} = \frac{b^2}{a^2}.\frac{ae\sqrt{2}}{\sqrt{e^2+1}}.\frac{\sqrt{e^2+1}}{b\sqrt{e^2-1}}$

$= \frac{b}{a}e\sqrt{2}.\frac{1}{\sqrt{e^2-1}}$

for H: $b^2 = a^2(e^2-1)$

or $b = a\sqrt{e^2-1}$

so $\frac{dy}{dx} = \frac{a\sqrt{e^2-1}}{a}e\sqrt{2}.\frac{1}{\sqrt{e^2-1}}$

$= e\sqrt{2}$.

at P on E: $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$

or $\frac{x^2}{a^2e^2} + \frac{y^2}{b^2} = 1$

Marking Guidelines

① working to $\frac{1}{e}$.

① working for simultaneous eqns.

① correct values of x, y at P

① correct use of $b^2 = a^2(e^2-1)$

① tangent gradient at P on H.

Comments.

· most did not find the point P – or trie[d] to give it generic co-ordinates.

· most who got this far did not use this to get expressions in ter[ms] of 'e'.

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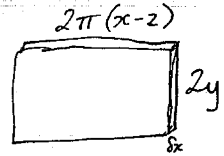
	Marking Guidelines	Comments
8) b) ii) (cont)		
$\therefore \frac{2x}{a^2e^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$		
$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2e^2}\cdot\frac{x}{y}$		
at P: $\frac{dy}{dx} = -\frac{b^2}{a^2e^2}\cdot\frac{ae\sqrt{2}}{\sqrt{e^2+1}}\cdot\frac{\sqrt{e^2+1}}{b\sqrt{e^2-1}}$		
$= -\frac{b\sqrt{2}}{ae}\cdot\frac{1}{\sqrt{e^2-1}}$		
$= -\frac{\sqrt{2}}{e}\cdot\frac{a\sqrt{e^2-1}}{a}\cdot\frac{1}{\sqrt{e^2-1}}$		
$= -\sqrt{2}\,\frac{1}{e}$.	① tangent gradient at P on E.	
ie $m_1 = e\sqrt{2}$ $\quad m_2 = -\sqrt{2}\cdot\frac{1}{e}$		· whilst many recognised the need to use this, the lack of co-ords for P in terms of 'e' made this impossible to work out.
$\therefore \tan\alpha = \left\lvert\frac{\sqrt{2}e - \left(-\sqrt{2}\frac{1}{e}\right)}{1+\sqrt{2}e\left(-\sqrt{2}\cdot\frac{1}{e}\right)}\right\rvert$	① correct working in angle between two lines formula.	
$= \left\lvert\frac{\sqrt{2}\left(e+\frac{1}{e}\right)}{1-2}\right\rvert$		
$= \sqrt{2}\left(e+\frac{1}{e}\right)$ //		

Alternative "Shells" method solution to Q8 (a)



$y = \sqrt{16-x^2}$

$\delta V = 2\pi\cdot 2y\,(x-2)\,\delta x$

$= 4\pi y\,(x-2)\,\delta x$ · note radius expression if using circle, centered at O.

$= 4\pi\,(x-2)\sqrt{16-x^2}\,\delta x$ · note limits, given use of ↑.

$\therefore V = 4\pi\int_2^4 (x-2)\sqrt{16-x^2}\,dx$ $\quad x = 4\sin\theta$, $dx = 4\cos\theta\,d\theta$ $\quad x=2\ \theta=\frac{\pi}{6}$, $x=4\ \theta=\frac{\pi}{2}$

$= 4\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (4\sin\theta-2)\sqrt{16-16\sin^2\theta}\,.\,4\cos\theta\,d\theta$

$= 16\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2(2\sin\theta-1)\,4\cos\theta\cos\theta\,d\theta$

$= 128\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin\theta-1)\cos^2\theta\,d\theta$

$= 128\pi\int 2\sin\theta\cos^2\theta - \cos^2\theta\,d\theta$

$= 128\pi\int 2\sin\theta\cos^2\theta - \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta$

$= 128\pi\left[-\frac{2}{3}\cos^3\theta - \frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$

$= 128\pi\left[0 - \frac{\pi}{4} - 0 - \left(-\frac{2}{3}\left(\frac{\sqrt{3}}{2}\right)^3 - \frac{\pi}{12} - \frac{1}{4}\frac{\sqrt{3}}{2}\right)\right]$

$= 128\pi\left[-\frac{\pi}{4} + \frac{\sqrt{3}}{4} + \frac{\pi}{12} + \frac{\sqrt{3}}{8}\right]$

$= 128\pi\left[\frac{3\sqrt{3}}{8} - \frac{\pi}{6}\right]$

$= 64\pi\left[\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right]$ ✓